LINEAR DEPENDENCY OF TRANSLATIONS AND SQUARE INTEGRABLE REPRESENTATIONS

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ABSTRACT. Let G be a locally compact group. We examine the problem of determining when nonzero functions in $L^2(G)$ have linearly independent left translations. In particular, we establish some results for the case when G has an irreducible, square integrable, unitary representation. We apply these results to the special cases of the affine group, the shearlet group and the Weyl-Heisenberg group. We also investigate the case when G has an abelian, closed subgroup of finite index.

1. INTRODUCTION

Let G be a locally compact Hausdorff group with left invariant Haar measure μ. Denote by $L^p(G)$ the set of complex-valued functions on G that are p-integrable with respect to μ, where $1 < p \in \mathbb{R}$. As usual, identify functions in $L^p(G)$ that differ only on a set of μ-measure zero. We shall write $\|\cdot\|_p$ to indicate the usual L^p-norm on $L^p(G)$. The *regular representation* of G on $L^p(G)$ is given by $L(g)f(x) = f(g^{-1}x)$, where $g, x \in G$ and $f \in L^p(G)$. The function $L(g)f$ is known as the left translation of f by g (many papers use the word "translate" instead of "translation"). In [22] Rosenblatt investigated the problem of determining when the left translations of a nonzero function f in $L^2(G)$ are linearly independent. In other words, when can there be a nonzero function $f \in L^2(G)$, some nonzero complex constants c_k, and distinct elements $g_k \in G$, where $1 \le k \le n, k \in \mathbb{N}$ (the positive integers) such that

$$\sum_{k=1}^{n} c_k L(g_k) f = 0? \tag{1.1}$$

It was shown in the introduction of [22] that if G has a nontrivial element of finite order, then there is a nonzero element in $L^2(G)$ that has a linear dependency among its left translations. Thus, when trying to find nontrivial functions that satisfy (1.1) it is more interesting to consider groups for which all nonidentity elements have infinite order. In the case of $G = \mathbb{R}^n$ it is known that every nonzero function in $L^2(\mathbb{R}^n)$ has no linear dependency among its left translations. Rosenblatt attacked (1.1) by trying to determine if there is a relationship between the linear independence of the translations of functions in $L^2(G)$ and the linear independence of an element and its images under the action of G in an irreducible representation of G. In order to gain insights into possible connections between these concepts, he computed examples for specific groups. The particular groups that he studied

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in [22] were the Heisenberg group and the affine group. What made these groups appealing is that they have irreducible representations that are intimately related to a time-frequency equation. Recall that an equation of the form

$$\sum_{k=1}^{n} c_k \exp(ib_k h(t)) f(a_k + t) = 0, \tag{1.2}$$

is a time-frequency equation, where $a_k, b_k \in \mathbb{R}, f \in L^2(\mathbb{R})$ and $h\colon \mathbb{R} \to \mathbb{R}$ is a nontrivial function. The case $h(t) = t$ corresponds to the Heisenberg group and $h(t) = e^t$ corresponds to the affine group.

Now suppose that G is a group that has an irreducible representation related to (1.2). Rosenblatt wondered if there existed a nontrivial $f \in L^2(\mathbb{R})$ that satisfied equation (1.2), then this f could be used to produce a nonzero $F \in L^2(G)$ with a linear dependency among its left translations. He then showed [22, Proposition 3.1] that there exists a nonzero $f \in L^2(\mathbb{R})$ that satisfies the following time-frequency equation

$$Cf(t) = f(t - \log 2) + \exp(-\frac{i}{2}e^t) f(t - \log 2), \tag{1.3}$$

where C is a constant. This time-frequency equation corresponds to the affine group A case since $h(t) = e^t$. This offers some hope that there might be a nonzero function in $L^2(A)$ that has a linear dependency among its left translations. However, there is no clear principle that can be used to show the existence of such a function given a nontrivial f that satisfies (1.3). Using the proof of the existence of f that satisfies (1.3) as a guide, a nonzero F in $L^2(A)$ with linearly dependent left translations was shown to exist [22, Proposition 3.2].

Even less is known about the Heisenberg group $H_n, n \in \mathbb{N}$. The relevant time-frequency equation, which has been intensely studied in the context of Gabor analysis, is

$$\sum_{k=1}^{m} c_k e^{2\pi i b_k \cdot t} f(t + a_k) = 0, \tag{1.4}$$

where c_k are nonzero constants, $a_k, b_k \in \mathbb{R}^n$, and $f \in L^2(\mathbb{R}^n)$. Linnell showed that $f = 0$ is the only solution to (1.4) when the subgroup generated by (a_k, b_k), where $k = 1, \dots, m$, is discrete. This gave a partial answer to a conjecture posed by Heil, Ramanathan and Topiwala on page 2790 of [11] that $f = 0$ is the only solution to (1.4) when $n = 1$. As far as we know the conjecture is still open.

The motivation for this paper is to give a clearer picture of the link between the linear independence of an element and its images under the action of G in an irreducible representation of G and the linear independence of the left translations of a function in $L^2(G)$. In Section 2 we will prove the following:

Proposition 1.5. *Let G be a locally compact group and suppose π is an irreducible, unitary, square integrable representation of G on a Hilbert space $\mathcal{H}_\pi$. If there exists a nonzero v in $\mathcal{H}_\pi$ such that*

$$\sum_{k=1}^{n} c_k \pi(g_k) v = 0$$

for some nonzero constants $c_k \in \mathbb{C}$ and $g_k \in G$, then there exists a nonzero $F \in L^2(G)$ that satisfies

$$\sum_{k=1}^{n} c_k L(g_k) F = 0.$$

In particular if there exists a nonzero v in $\mathcal{H}_\pi$ with linearly dependent translations, then there exists a nonzero F in $L^2(G)$ with linearly dependent translations.

In Section 3 we will use Proposition 1.5 to construct explicit examples of nontrivial functions in $L^2(A)$, where A is the affine group, that have a linear dependency among their left translations.

In Section 4 we investigate the case where G is a discrete group. We will see that the problem of determining if the left translations of a nonzero function in $\ell^2(G)$ forms a linearly independent set is related to the strong Atiyah conjecture. We shall briefly review the strong Atiyah conjecture in Section 4.

After considering the discrete group case in Section 4, we shall return to studying the linear dependency problem for groups that satisfy our original hypotheses. Let K be a subgroup of a group G. If $k \in K, x \in G$, and $f \in L^2(G)$, then we shall say that

$$L(k)f(x) = f(k^{-1}x)$$

is a *left K-translation* of f. In Section 5 we shall prove

Theorem 1.6. *Let G be a locally compact, σ-compact group and let K be a torsion-free discrete subgroup of G. If K satisfies the strong Atiyah conjecture, then each nonzero function in $L^2(G)$ has linearly independent K-translations.*

In Section 6 we shall study the Weyl-Heisenberg group $\tilde{H}_n$, a variant of the Heisenberg group, H_n. The group $\tilde{H}_n$ is of interest to us because it has an irreducible unitary representation on $L^2(\mathbb{R}^n)$, the Schrödinger representation, which is square integrable. Furthermore, the time-frequency equation (1.4) is related to the Schrödinger representation. Now if K is a torsion-free discrete subgroup of $\tilde{H}_n$, then by Theorem 1.6 every nonzero element in $L^2(\tilde{H}_n)$ has linearly independent left K-translations. It will then follow from Proposition 1.5 that if the subgroup of $\mathbb{R}^{2n}$ generated by $(a_k, b_k), 1 \le k \le m$, is discrete and the product $a_h \cdot b_k \in \mathbb{Q}$ for all h, k, then $f = 0$ is the only solution to (1.4), see Proposition 6.3. This gives a new proof of a special case of [16, Proposition 1.3] and sheds new insights on the problem.

In Section 7 we consider the problem of determining the linear independence of the left translations of a function in $L^2(S)$, where S is the shearlet group. By using Proposition 1.5 we will see that this problem is related to the question of determining the linear independence of a shearlet system of a function in $L^2(\mathbb{R}^2)$, which was recently studied in [19].

In the last section of the paper we investigate the linear independence of left translations of functions in $L^p(G)$ for virtually abelian groups G with no nontrivial compact subgroups. In particular, we generalize [5, Theorem 1.2].

2. Proof of Proposition 1.5

In this section we will prove Proposition 1.5. Before we give our proof we will give some necessary definitions. A *unitary representation* of G is a homomorphism π from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on a nonzero Hilbert space

$\mathcal{H}_\pi$ that is continuous with respect to the strong operator topology. This means that $\pi\colon G \to U(\mathcal{H}_\pi)$ satisfies $\pi(xy) = \pi(x)\pi(y), \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, and $x \to \pi(x)u$ is continuous from G to $\mathcal{H}_\pi$ for each $u \in \mathcal{H}_\pi$. A closed subspace W of $\mathcal{H}_\pi$ is said to be *invariant* if $\pi(x)W \subseteq W$ for all $x \in G$. If the only invariant subspaces of $\mathcal{H}_\pi$ are $\mathcal{H}_\pi$ and 0, then π is said to be an *irreducible representation* of G. A representation that is not irreducible is defined to be a *reducible representation*. If π_1 and π_2 are unitary representations of G, an *intertwining operator* for π_1 and π_2 is a bounded liner map $T\colon \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ that satisfies $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$. We will assume throughout this paper that the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_\pi$ is conjugate linear in the second component. If $u, v \in \mathcal{H}_\pi$, a *matrix coefficient* of π is the function $F_{v,u}\colon G \to \mathbb{C}$ defined by

$$F_{v,u}(x) = \langle v, \pi(x)u \rangle.$$

We will indicate the $F_{u,u}$ case by F_u. An irreducible representation π is said to be *square integrable* if there exists a nonzero $u \in \mathcal{H}_\pi$ such that $F_u \in L^2(G)$. We shall say that $u \in \mathcal{H}_\pi$ is *admissible* if $F_u \in L^2(G)$. The set of admissible elements in $\mathcal{H}_\pi$ will be denoted by $\mathrm{Ad}(\mathcal{H}_\pi)$. A consequence of π being irreducible is that if there is a nonzero admissible element in $\mathcal{H}_\pi$, then $\mathrm{Ad}(\mathcal{H}_\pi)$ is dense in $\mathcal{H}_\pi$. In fact, $\mathrm{Ad}(\mathcal{H}_\pi) = \mathcal{H}_\pi$ if G is unimodular, in addition to $\mathrm{Ad}(\mathcal{H}_\pi)$ containing a nonzero element. By [10, Theorem 3.1] there exists a self adjoint positive operator $C\colon \mathrm{Ad}(\mathcal{H}_\pi) \to \mathcal{H}_\pi$ such that if $u \in \mathrm{Ad}(\mathcal{H}_\pi)$ and $v \in \mathcal{H}_\pi$, then

$$\begin{aligned}\int_G |\langle v, \pi(x)u\rangle|^2 \, d\mu &= \int_G \langle v, \pi(x)u \rangle \overline{\langle v, \pi(x)u \rangle} \, d\mu \\ &= \|Cu\|^2 \|v\|^2,\end{aligned}$$

where $\| \cdot \|$ denotes the $\mathcal{H}_\pi$-norm. Thus if $u \in \mathrm{Ad}(\mathcal{H}_\pi)$, then $F_{v,u} \in L^2(G)$ for all $v \in \mathcal{H}_\pi$.

We now prove Proposition 1.5. Suppose there exists a nonzero $v \in \mathcal{H}_\pi$ for which there exists a linear dependency among some of the elements $\pi(g)v$, where $g \in G$. So there exists nonzero constants $c_1, c_2, \dots, c_n$ and elements $g_1, g_2, \dots, g_n$ from G with $\pi(g_j) \neq \pi(g_k)$ if $j \neq k$ such that

$$\sum_{k=1}^{n} c_k \pi(g_k) v = 0. \tag{2.1}$$

Let $u \in \mathrm{Ad}(\mathcal{H}_\pi)$. Then $0 \neq F_{v,u} \in L^2(G)$. Since π is unitary, $\langle \pi(g)v, \pi(x)u\rangle = \langle v, \pi(g^{-1}x)u \rangle$ for all x and g in G; in other words the continuous linear map $v \mapsto F_{v,u}\colon \mathcal{H} \to L^2(G)$ intertwines π with the regular representation L. Combining this observation with our hypothesis (2.1) yields for all $x \in G$,

$$\sum_{k=1}^{n} c_k L(g_k) F_{v,u}(x) = 0,$$

that is $F_{v,u}$ has linearly dependent left translations. The proof of Proposition 1.5 is now complete.

3. The Affine Group

In this section we give examples of nonzero functions in $L^2(G)$, where G is the affine group, that have a linear dependency among some of its left translations. Let $\mathbb{R}$ denote the real numbers and let $\mathbb{R}^*$ be the set $\mathbb{R} \setminus \{0\}$. Recall that $\mathbb{R}$ is a group

under addition and $\mathbb{R}^*$ is a group with respect to multiplication. The affine group, also known as the $ax+b$ group, is defined to be the semidirect product of $\mathbb{R}^*$ and $\mathbb{R}$. That is,

$$G = \mathbb{R}^* \rtimes \mathbb{R}.$$

Let (a,b) and (c,d) be elements of G. The group operation on G is given by $(a,b)(c,d) = (ac, b+ad)$. The identity element of G is $(1,0)$ and $(a,b)^{-1} = (a^{-1}, -a^{-1}b)$. The left Haar measure on G is $d\mu = \frac{dadb}{a^2}$ and the right Haar measure is $d\mu = \frac{dadb}{|a|}$. Thus G is a nonunimodular group because the right and left Haar measures do not agree. So $f \in L^2(G)$ if and only if

$$\int_{\mathbb{R}} \int_{\mathbb{R}^*} |f(a,b)|^2 \frac{dadb}{a^2} < \infty.$$

An irreducible unitary representation of G can be defined on $L^2(\mathbb{R})$ by

$$\pi(a,b)f(x) = |a|^{-1/2} f\left(\frac{x-b}{a}\right),$$

where $(a,b) \in G$ and $f \in L^2(\mathbb{R})$. Before we show that π is square integrable we recall some facts from Fourier analysis.

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform of f is defined to be

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx,$$

where $\xi \in \mathbb{R}$. The Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R})$. For $y \in \mathbb{R}$ we also have the following unitary operators on $L^2(\mathbb{R})$,

$$T_y f(x) = f(x-y), E_y f(x) = e^{2\pi i y x} f(x)$$

$$D_y f(x) = |y|^{-1/2} f\left(\frac{x}{y}\right), (y \neq 0).$$

Given $f, g \in L^2(\mathbb{R})$ the following relations are also true $\langle f, T_y g\rangle = \langle T_{-y} f, g\rangle$, $\langle f, E_y g\rangle = \langle E_{-y} f, g\rangle$ and $\langle f, D_y g\rangle = \langle D_{y^{-1}} f, g\rangle$. Furthermore, $\widehat{T_y f} = E_{-y}\widehat{f}$ and $\widehat{D_y f} = D_{y^{-1}}\widehat{f}$. Observe that for $(a,b) \in G$ and $f \in L^2(\mathbb{R})$,

$$\pi(a,b)f(x) = T_b D_a f(x) = |a|^{-1/2} f\left(\frac{x-b}{a}\right).$$

Using the above relations it can be shown that for $f \in L^2(\mathbb{R})$,

$$\int_G \langle f, \pi(a,b) f\rangle\, d\mu = \int_{\mathbb{R}} \int_{\mathbb{R}^*} |\langle f, T_b D_a f\rangle|^2 \frac{dadb}{a^2} = \|f\|_2^2 \int_{\mathbb{R}^*} \frac{|\widehat{f}(\xi)|^2}{|\xi|} d\xi,$$

see [12, Theorem 3.3.5]. Thus $f \in L^2(\mathbb{R})$ is admissible if $\int_{\mathbb{R}^*} \frac{|\widehat{f}(\xi)|^2}{|\xi|} d\xi < \infty$. The function $f(x) = \sqrt{2\pi} x e^{-\pi x^2}$ satisfies this criterion since $\widehat{f}(\xi) = -\sqrt{2\pi} i \xi e^{-\pi \xi^2}$, which we obtained by combining [20, Proposition 2.2.5] with [20, Example 2.2.7]. Hence π is a square integrable, irreducible unitary representation of the affine group G. We are now ready to construct a nonzero function in $L^2(G)$ that has linearly dependent left translations.

Let $\chi_{[0,1)}$ be the characteristic function on the interval $[0,1)$. It follows from the following *refinement equation*

$$\chi_{[0,1)}(x) = \chi_{[0,1)}(2x) + \chi_{[0,1)}(2x-1)$$

that

$$\pi(1,0)\chi_{[0,1)}(x) = 2^{-1/2}\pi\left(2^{-1},0\right)\chi_{[0,1)}(x) + 2^{-1/2}\pi\left(2^{-1},2^{-1}\right)\chi_{[0,1)}(x). \tag{3.1}$$

Thus $\chi_{[0,1)}$ has a linear dependency among the $\pi(a,b)\chi_{[0,1)}$, where $(a,b) \in G$. We now use $\chi_{[0,1)}$ to construct a nontrivial function in $L^2(G)$ that has linearly dependent left translations. Let $f \in L^2(\mathbb{R})$ be an admissible function for π and let $(a,b) \in G$. Then the function

$$F(a,b) = \langle \chi_{[0,1)}, \pi(a,b)f\rangle = \int_0^1 |a|^{-1/2}\overline{f\left(\frac{x-b}{a}\right)}\,dx$$

belongs to $L^2(G)$. By Proposition 1.5, $F(a,b)$ has linearly dependent left translations. More specifically,

$$L(1,0)F(a,b) = 2^{-1/2}L(2^{-1},0)F(a,b) + 2^{-1/2}L(2^{-1},2^{-1})F(a,b),$$

which translates to

$$\int_0^1 |a|^{-1/2}\overline{f\left(\frac{x-b}{a}\right)}\,dx = \int_0^{1/2} |a|^{-1/2}\overline{f\left(\frac{x-b}{a}\right)}\,dx + \int_{1/2}^1 |a|^{-1/2}\overline{f\left(\frac{x-b}{a}\right)}\,dx.$$

Equation (3.1) above was used in the proof of [22, Proposition 3.1] to show that the time-frequency equation (1.3) with $C = \sqrt{2}$ has a nonzero solution. Basically, equation (1.3) is a reinterpretation of the above refinement equation where the representation π is replaced by an equivalent representation. See [22, Section 3] for the details.

We now turn our attention to the subgroup K of the affine group G which consists of all $(a,b) \in G$ for which $a > 0$. This was the version of the affine group considered in [22]. The left Haar measure for K is the same as the left Haar measure for G. Up to unitary equivalence there are two irreducible unitary infinite dimensional representations of K, see [7, Section 6.7] for the details. One of these representations is given by

$$\pi^+(a,b)f(x) = a^{1/2}e^{2\pi ibx}f(ax) = E_bD_{a^{-1}}f(x),$$

where $(a,b) \in K$ and $f \in L^2(0,\infty)$. The representation π^+ is square integrable. We are now ready to produce a nontrivial function in $L^2(K)$ that has linearly dependent left translations. From (3.1) we have

$$\chi_{[0,1)} = 2^{-1/2}D_{2^{-1}}\chi_{[0,1)} + 2^{-1/2}T_{2^{-1}}D_{2^{-1}}\chi_{[0,1)}.$$

By taking Fourier transforms we obtain

$$\begin{aligned}\widehat{\chi}_{[0,1)}(\xi) &= 2^{-1/2}D_2\widehat{\chi}_{[0,1)}(\xi) + 2^{-1/2}E_{-2^{-1}}D_2\widehat{\chi}_{[0,1)}(\xi)\\ &= 2^{-1/2}\pi^+(2^{-1},0)\widehat{\chi}_{[0,1)}(\xi) + 2^{-1/2}\pi^+(2^{-1},-2^{-1})\widehat{\chi}_{[0,1)}(\xi).\end{aligned}$$

Hence, there is a linear dependency among the $\pi^+(a,b)\widehat{\chi}_{[0,1)}$, where $(a,b) \in K$. It follows from

$$\widehat{\chi}_{[0,1)}(\xi) = \frac{e^{-2\pi i\xi} - 1}{-2\pi i\xi}$$

that $\widehat{\chi}_{[0,1)} \in L^2(0,\infty)$. Pick an admissible function $f \in L^2(0,\infty)$ for π^+. Then the function

$$F(a,b) = \langle \widehat{\chi}_{[0,1)}, \pi^+(a,b)f\rangle = \int_0^\infty \widehat{\chi}_{[0,1)}(\xi)a^{1/2}e^{-2\pi ib\xi}\overline{f(\xi)}\,d\xi$$

is a member of $L^2(K)$. Proposition 1.5 yields the following linear dependency in $L^2(K)$ among the left translations of $F(a,b)$.

$$F(a,b) = 2^{-1/2}L(2^{-1},0)F(a,b) + 2^{-1/2}L(2^{-1},-2^{-1})F(a,b).$$

This equation can easily be verified by using the relations

$$\widehat{\chi}_{[0,1)}(\frac{\xi}{2})(1+e^{-\pi i\xi}) = \frac{(e^{-\pi i\xi}-1)(1+e^{-\pi i\xi})}{-2\pi i\xi} = \widehat{\chi}_{[0,1)}(\xi).$$

4. Discrete groups and the Atiyah conjecture

In this section we connect the problem of linear independence of left translations of a function to the Atiyah conjecture. Unless otherwise stated we make the assumption that all groups in this section are discrete. For discrete groups the Haar measure is counting measure. Let f be a complex-valued function on a group G. We will represent f as a formal sum $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{C}$ and $f(g) = a_g$. Denote by $\ell^2(G)$ those formal sums for which $\sum_{g\in G} |a_g|^2 < \infty$, and $\mathbb{C}G$, the group ring of G over $\mathbb{C}$ will consist of all formal sums that satisfy $a_g = 0$ for all but finitely many g. The group ring $\mathbb{C}G$ can also be thought of as the set of all functions on G with compact support and $\ell^2(G)$ is a Hilbert space with Hilbert basis $\{g \mid g \in G\}$. If $g \in G$ and $f = \sum_{x\in G} a_x x \in \ell^2(G)$, then the left translation of f by g is represented by the formal sum $\sum_{x\in G} a_{g^{-1}x} x$ since $L(g)f(x) = f(g^{-1}x)$. Suppose $\alpha = \sum_{g\in G} a_g g \in \mathbb{C}G$ and $f = \sum_{g\in G} b_g g \in \ell^2(G)$. We define a multiplication, known as convolution, $\mathbb{C}G \times \ell^2(G) \to \ell^2(G)$ by

$$\alpha * f = \sum_{g,h\in G} a_g b_h gh = \sum_{g\in G}\left(\sum_{h\in G} a_{gh^{-1}} b_h\right) g.$$

Sometimes we will write αf instead of $\alpha * f$. Left multiplication by an element of $\mathbb{C}G$ is a bounded linear operator on $\ell^2(G)$. So $\mathbb{C}G$ can be considered as a subring of $\mathcal{B}(\ell^2(G))$, the space of bounded linear operators on $\ell^2(G)$. We shall say that G is torsion-free if the only element of finite order in G is the identity element of G. The strong Atiyah conjecture for the group G is concerned with the values the L^2-Betti numbers can take, and it implies the following conjecture, which can also be considered an analytic version of the zero divisor conjecture.

Conjecture 4.1. *Let G be a torsion-free group. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq f \in \ell^2(G)$, then $\alpha * f \neq 0$.*

The hypothesis that G is torsion-free is essential. Indeed, let 1 be the identity element of G and let $g \in G$ such that $g \neq 1$ and $g^n = 1$ for some $n \in \mathbb{N}$. Then $(1 + g + \cdots + g^{n-1}) * (1 - g) = 0$. The Atiyah conjecture is important in the study of von Neumann dimension. For further information see [3, 14, 15] and [18, Section 10]. In particular, the assertion of Conjecture 4.1 is known for free groups, left-ordered groups and elementary amenable groups.

The following proposition gives the link between zero divisors and the linear independence of left translations of a function.

Proposition 4.2. *Let G be a discrete group and let $f \in \ell^2(G)$. Then f has linearly independent left translations if and only if $\alpha * f \neq 0$ for all nonzero $\alpha \in \mathbb{C}G$.*

Proof. Let $g \in G$ and let $f = \sum_{x \in G} a_x x \in \ell^2(G)$. Then

$$g * f = \sum_{x \in G} a_x g x = \sum_{x \in G} a_{g^{-1}x} x = L(g) f.$$

Consequently, if $g_1, \dots, g_n \in G$ are distinct and $c_1, \dots, c_n$ are constants, then

$$\sum_{k=1}^{n} c_k L(g_k) f = \sum_{k=1}^{n} c_k g_k * f = \left(\sum_{k=1}^{n} c_k g_k \right) * f.$$

The proposition now follows since $\sum_{k=1}^{n} c_k g_k \in \mathbb{C}G$. □

As we saw in Section 3 there are nontrivial, square integrable functions on the affine group that have a linear dependency among their left translations. Since all nonidentity elements of the affine group have infinite order, it seems reasonable by taking a discrete subgroup D of the affine group, such as $1 \rtimes \mathbb{Z}$, we might be able to construct a nontrivial function in $\ell^2(D)$ that has a linear dependency among its left translations. It would then be an immediate consequence of Proposition 4.2 that Conjecture 4.1 is false. However, for discrete subgroups D of the affine group it is not true that there exists a nonzero function in $\ell^2(D)$ with a linear dependency among its left translations. Indeed, the affine group is a solvable Lie group, and all discrete subgroups of solvable Lie groups are polycyclic. By [14, Theorem 2] Conjecture 4.1 is true for torsion-free elementary amenable groups, a class of groups that contain all torsion-free polycyclic groups.

5. Proof of Theorem 1.6

In this section we prove Theorem 1.6. Recall that our standing assumptions on the group G is that it is locally compact, Hausdorff with left invariant Haar measure μ. Let $g_1, \dots, g_n$ be elements of G and let $c_1, \dots, c_n \in \mathbb{C}$ be some constants. Set $\theta = \sum_{k=1}^{n} c_k L(g_k)$. So $\theta \in \mathcal{B}(L^2(G))$, the set of bounded linear operators on $L^2(G)$. Define

$$\mathbb{C}G = \{ \sum_{g \in G} a_g L(g) \mid a_g = 0 \text{ for all but finitely many } g \in G \}.$$

Note that there exists a nonzero $f \in L^2(G)$ with linearly dependent left translations if and only if there exists a nonzero $\theta \in \mathbb{C}G$ with $\theta f = 0$.

For the rest of this section H will denote a discrete subgroup of G. We will also assume that G is σ-compact in addition to our standing assumptions on G. The subgroup H acts on G by left multiplication. By [1, Proposition B.2.4] there exists a Borel fundamental domain for this action of H on G. More precisely, there exists a Borel subset B of G such that $hB \cap B = \emptyset$ for all $h \in H \setminus 1$ and $G = HB$ (thus B is a system of right coset representatives of H in G which is also a Borel subset). If X is a Borel subset of G, then we will identify $L^2(X)$ with the subspace of $L^2(G)$ consisting of all functions on G whose support is contained in X.

Let $\{q_i \mid i \in \mathcal{I}\}$ be a Hilbert basis for $L^2(B)$. We claim that $S := \{L(h)q_i \mid h \in H,\ i \in \mathcal{I}\}$ is a Hilbert basis for $L^2(G)$. First we show that S is orthonormal. Write hq_i for $L(h)q_i$. If $h \neq k$, then $\langle hq_i, kq_j \rangle = 0$, because the supports of hq_i and hq_j are contained in hB and kB respectively, which are disjoint subsets. On the other hand if $h = k$, then $\langle hq_i, hq_j \rangle = \langle q_i, q_j \rangle$, because the Haar measure is left invariant. This proves that S is orthonormal. Finally we show that the closure of the linear span $\overline{S}$ of S is $L^2(G)$. Denote by χ_{hB} the characteristic function on hB. If $f \in L^2(G)$, then we may write $f = \sum_{h \in H} f_h$, where $f_h = \chi_{hB} f$ (so f_h

has support contained in hB). Thus it will be sufficient to show that $L^2(hB) \subseteq \overline{S}$. Since $\overline{S}$ is invariant under H, it will be sufficient to show that $L^2(B) \subseteq \overline{S}$, which is obvious because the q_i form a Hilbert basis for $L^2(B)$.

For $i \in \mathcal{I}$, let $S_i = \{L(h)q_i \mid h \in H\}$ and let $\overline{S}_i$ denote the closure of the linear span of S_i. Now $L^2(G) = \bigoplus_{i\in\mathcal{I}} \overline{S}_i$ (where $\bigoplus$ indicates the Hilbert direct sum). The spaces $\overline{S}_i$ are isometric to $\ell^2(H)$. Indeed, define a map T_i from the Hilbert basis S_i of $\overline{S}_i$ to the Hilbert basis H of $\ell^2(H)$ via $L(h)q_i \mapsto h$. Extend T_i linearly to obtain an isometry $T_i \colon \overline{S}_i \to \ell^2(H)$. Moreover, the isometry T_i intertwines the natural left actions of H on $\overline{S}_i$ and $\ell^2(H)$. Also let π_i denote the projection of $L^2(G)$ onto $\overline{S}_i$. Then π_i also intertwines the natural left actions of H on $L^2(G)$ and $\overline{S}_i$. Now suppose that there exists a nonzero $f \in L^2(G)$ and a nonzero $\theta \in \mathbb{C}H$ that satisfies $\theta f = 0$. Then $k := T_i\pi_i f \neq 0$ for some i, and $\theta * k = \theta k = 0$ because $T_i\pi_i$ commutes with $\mathbb{C}H$. Furthermore, $k \in l^2(H)$. We can summarize the above as follows:

Proposition 5.1. *Let H be a discrete subgroup of the σ-compact locally compact group G and let $\theta \in \mathbb{C}H$. If $\theta f = 0$ for some nonzero $f \in L^2(G)$, then $\theta * k = 0$ for some nonzero $k \in \ell^2(H)$.*

Now let H be a torsion-free group which satisfies the strong Atiyah conjecture, e.g. a torsion-free elementary amenable group. Then for $0 \neq \theta \in \mathbb{C}H$, we know that $\theta * k \neq 0$ for all non-zero $k \in \ell^2(H)$. It follows from Proposition 5.1 that $\theta f \neq 0$ for all nonzero $f \in L^2(G)$, in other words, any nonzero element of $L^2(G)$ has linearly independent H-translations. The proof of Theorem 1.6 is now complete.

In a similar fashion, we can prove

Theorem 5.2. *Let G be a locally compact σ-compact group and let H be an amenable discrete subgroup of G. If α is a non-zerodivisor in $\mathbb{C}H$, then $\alpha * f \neq 0$ for all nonzero $f \in L^2(G)$.*

Proof. Since $\alpha\beta \neq 0$ for all nonzero $\beta \in \mathbb{C}H$, it follows that $\alpha\beta \neq 0$ for all nonzero $\beta \in \ell^2(H)$ by [6, Theorem] (or use [18, Theorem 6.37]). The result now follows from Proposition 5.1. $\square$

We saw in Section 3 that for the affine group A there exist nonzero f in $L^2(A)$ with linearly dependent left translations. However, $\mathbb{Z}$ can be identified with the discrete subgroup $1 \rtimes \mathbb{Z}$ of A. A direct consequence of Theorem 1.6 is

Corollary 5.3. *Let A be the affine group. Then every nonzero f in $L^2(A)$ has linearly independent left $\mathbb{Z}$-translations.*

As noted in Section 4, if H is a discrete group, we may regard $\mathbb{C}H$ as a subalgebra of $\mathcal{B}(\ell^2(H))$. Recall that the reduced group C^*-algebra of H, denoted $\mathrm{C}^*_{\mathrm{r}}(H)$, is the operator norm closure of $\mathbb{C}H$ in $\mathcal{B}(\ell^2(H))$, and the group von Neumann algebra of H, denoted $\mathcal{N}(H)$, is the weak closure of $\mathbb{C}H$ in $\mathcal{B}(\ell^2(H))$. We can also identify the norm and weak closures of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$ with $\mathrm{C}^*_{\mathrm{r}}(H)$ and $\mathcal{N}(H)$ respectively. Though this is not needed in the sequel, we hope it maybe useful to record this.

For $\theta \in \mathcal{B}(L^2(G))$ or $\mathcal{B}(\ell^2(H))$, let $\|\theta\|$ or $\|\theta\|'$ denote the corresponding operator norms respectively. We retain the notation used in the proof of Proposition 5.1. Observe that we have a natural isomorphism $\mathcal{B}(\overline{S}_i) \to \mathcal{B}(\ell^2(H))$ induced by T_i. Furthermore $L^2(G) = \bigoplus_{i\in\mathcal{I}} \overline{S}_i$ (where $\bigoplus$ indicates the Hilbert direct sum), and this a decomposition as left $\mathbb{C}H$-modules. We will need the following:

Lemma 5.4. *Let $\theta \in \mathbb{C}H$. Then $\|\theta\| = \|\theta\|'$*

Proof. Note that θ can be considered as an operator on $L^2(G)$ or $\ell^2(H)$. If $u \in L^2(G)$, we may write $u = \sum_{i\in\mathcal{I}} u_i$ with $u_i \in \overline{S}_i$, so

$$\begin{aligned}\|\theta\| = \sup_{u\in L^2(G),\ \|u\|_2=1} \|\theta u\|_2 &= \sup_{u\in L^2(G),\ \|u\|_2=1} \|\theta \sum_{i\in\mathcal{I}} u_i\|_2 \\ &\le \sup_{u\in L^2(G),\ \|u\|_2=1} \sqrt{\sum_{i\in\mathcal{I}} \|\theta\|'^2 \|u_i\|_2^2} = \|\theta\|'.\end{aligned}$$

Fix $\iota \in \mathcal{I}$. Then

$$\|\theta\|' = \sup_{u\in\overline{S}_\iota,\ \|u\|_2=1} \|\theta u\|_2 \le \sup_{u\in L^2(G),\ \|u\|_2=1} \|\theta u\|_2 \le \|\theta\|.$$

Therefore, $\|\theta\| = \|\theta\|'$. □

Denote by $\mathcal{O}(H)$ the operator norm closure, and $\mathcal{W}(H)$ the weak closure of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$. The space $\mathcal{W}(H)$ is a von Neumann algebra and by the double commutant theorem is equal to the strong closure of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$. Note that $\mathcal{O}(H) \subseteq \mathcal{W}(H)$ and $\mathrm{C}^*_\mathrm{r}(H) \subseteq \mathcal{N}(H)$. We now relate these various algebras:

Proposition 5.5. *There is a $*$-isomorphism $\alpha\colon \mathcal{W}(H) \to \mathcal{N}(H)$. Moreover, α preserves the operator norm and maps $\mathcal{O}(H)$ onto $\mathrm{C}^*_\mathrm{r}(H)$.*

Proof. Recall that for $u \in L^2(G)$, we can uniquely write $u = \sum_{i\in\mathcal{I}} u_i$ with $u_i \in \overline{S}_i$. Let $\theta \in \mathcal{W}(H)$. Then there exists a net (θ_i) in $\mathbb{C}H$ which converges strongly to θ. Therefore for every $u \in L^2(G)$, the net $(\theta_i u)$ is convergent in $L^2(G)$, consequently the net $(\theta_i u_j)$ is convergent for every j, in particular $(\theta_i f)$ is a Cauchy net in $\ell^2(H)$ for every $f \in \ell^2(H)$. We deduce that (θ_i) is a Cauchy net in $\mathcal{B}(\ell^2(H))$ (in the strong operator topology) and hence converges to an operator $\theta' \in \mathcal{N}(H)$. We note that θ' doesn't depend on the choice of the net (θ_i) and therefore we have a well-defined map $\alpha\colon \mathcal{W}(H) \to \mathcal{N}(H)$, where $\alpha(\theta) = \theta'$ and α is the identity on $\mathbb{C}H$.

We now construct the inverse to α by reversing the above steps. Let $\phi \in \mathcal{N}(H)$. By the Kaplansky density theorem there exists a net (θ_i) in $\mathbb{C}H$ which converges strongly to ϕ and $\|\theta_i\|'$ bounded. Thus $\|\theta_i\|$ is bounded because $\|\theta_i\| = \|\theta_i\|'$ for each i by Lemma 5.4. Now let $u \in L^2(G)$. If $\mathcal{J}$ is a finite subset of $\mathcal{I}$, set $v_\mathcal{J} = \sum_{j\in\mathcal{J}} u_j$. Then $(\theta_i v_j)$ converges in $L^2(G)$ for every $\mathcal{J}$. Since $\|\theta_i\|$ is bounded, it follows that $(\theta_i u)$ is convergent in $L^2(G)$ and we conclude that (θ_i) converges strongly to an operator $\tilde{\phi} \in \mathcal{B}(L^2(G))$. It follows that we have a well-defined map $\phi \to \tilde{\phi}\colon \mathcal{N}(H) \to \mathcal{W}(H)$, which is the inverse to α.

It is easily checked that α is a $*$-isomorphism and therefore is an isomorphism of C^*-algebras, in particular it preserves the operator norm. We deduce that α maps $\mathcal{O}(H)$ onto $\mathrm{C}^*_\mathrm{r}(H)$. □

Remark 5.6. *Proposition 5.5 can be used to give a different proof of Proposition 5.1; for details, see* [21, Chapter 2.5].

6. The Weyl-Heisenberg group

In this section we use techniques developed in this paper to determine when $f = 0$ is the only solution to the time-frequency equation (1.4). The relevant group

here is the Weyl-Heisenberg group since it has an irreducible representation that is square integrable.

Let $n \in \mathbb{N}$. The *Heisenberg group* H_n is the set of $(n+2) \times (n+2)$ matrices of the form

$$\begin{pmatrix} 1 & a & z \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a is a $1 \times n$ matrix, b is a $n \times 1$ matrix, the zero in the $(2,1)$ position is the $n \times 1$ zero matrix, the zero in the $(3,2)$ position is the $1 \times n$ zero matrix, and the 1_n in the $(2,2)$ position is the $n \times n$ identity matrix. Another way to represent H_n is as the product $\mathbb{R} \times \widehat{\mathbb{R}^n} \times \mathbb{R}^n$. Here we view $\mathbb{R}^n$ as $n \times 1$ column matrices and $\widehat{\mathbb{R}^n}$ as $1 \times n$ row matrices. For $(z_1, a_1, b_1), (z_2, a_2, b_2) \in H_n$ the group law becomes $(z_1, a_1, b_1)(z_2, a_2, b_2) = (z_1+z_2+a_1 \cdot b_2, a_1+a_2, b_1+b_2)$. Thus the identity element in H_n is $(0,0,0)$ and $(z,a,b)^{-1} = (a \cdot b - z, -a, -b)$. For $f \in L^2(\mathbb{R}^n)$ and $(z,a,b) \in H_n$ define

$$\pi(z,a,b)f(x) = e^{2\pi i z} e^{-2\pi i a \cdot b} e^{2\pi i a \cdot x} f(x-b).$$

It turns out that π is a representation of H_n on $L^2(\mathbb{R}^n)$. Indeed, let (z_1, a_1, b_1), $(z_2, a_2, b_2) \in H_n$. Then

$$\begin{aligned}
\pi(z_1, a_1, b_1)\big(\pi(z_2, a_2, b_2)f(x)\big) &= \pi(z_1, a_1, b_1)\big(e^{2\pi i z_2} e^{-2\pi i a_2 \cdot b_2} e^{2\pi i a_2 \cdot x} f(x-b_2)\big) \\
&= e^{2\pi i z_1} e^{2\pi i z_2} e^{-2\pi i a_1 \cdot b_1} e^{-2\pi i a_2 \cdot b_2} e^{2\pi i a_1 \cdot x} e^{2\pi i a_2 \cdot (x-b_1)} f(x-b_2-b_1) \\
&= e^{2\pi i (z_1+z_2)} e^{-2\pi i (a_1 \cdot b_1 + a_2 \cdot b_2)} e^{-2\pi i a_2 \cdot b_1} e^{2\pi i a_2 \cdot x} f(x-(b_1+b_2)) \\
&= e^{2\pi i (z_1+z_2+a_1 \cdot b_2)} e^{-2\pi i (a_1+a_2)\cdot(b_1+b_2)} e^{2\pi i (a_1+a_2)\cdot x} f(x-(b_1+b_2)) \\
&= \big(\pi(z_1, a_1, b_1)\pi(z_2, a_2, b_2)\big) f(x).
\end{aligned}$$

Let $Z = \langle(2\pi, 0, 0)\rangle$, the subgroup of H_n generated by $(2\pi, 0, 0)$. Set $\tilde{H}_n = H_n/Z$. The group $\tilde{H}_n$ is known as the *Weyl-Heisenberg group*. Clearly $Z = \ker \pi$ and so π induces a representation $\tilde{\pi}$ on $\tilde{H}_n$. Observe that $\tilde{H}_n = \{(t,a,b) \mid t \in \mathbb{T}, a, b \in \mathbb{R}^n\}$ (here $\mathbb{T}$ is the unit circle $\{z \in \mathbb{Z} \mid |z| = 1\}$). The Lebesgue measure on $H_n = \mathbb{R} \times \widehat{\mathbb{R}^n} \times \mathbb{R}^n$ is left and right invariant Haar measure on H_n. Similarly, Lebesgue measure on $\mathbb{T} \times \widehat{\mathbb{R}^n} \times \mathbb{R}^n$ is left and right invariant Haar measure on $\tilde{H}_n$ (here Lebesgue measure on $\mathbb{T}$ is normalized so that $\int_{\mathbb{T}} dt = 1$). The next result was proved in [12, Proposition 3.2.4] for the special case $n = 1$. By interchanging the roles of a and b the proof given there carries through verbatim to our case.

Proposition 6.1. *If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\int_{\widehat{\mathbb{R}^n}} \int_{\mathbb{R}^n} \int_{\mathbb{T}} |\langle f, \tilde{\pi}(t,a,b)g\rangle|^2 \, dt db da = \|f\|_2^2 \|g\|_2^2.$$

Corollary 6.2. *The representation $\tilde{\pi}$ of $\tilde{H}_n$ on $L^2(\mathbb{R}^n)$ is irreducible and square integrable, and every $g \in L^2(\mathbb{R}^n)$ is admissible.*

Proof. By taking $f = g$ in the above proposition we see immediately that every element of $L^2(\mathbb{R}^n)$ is admissible. Suppose $g \in L^2(\mathbb{R}^n) \setminus \{0\}$ is fixed and assume $f \in L^2(\mathbb{R}^n)$ satisfies $\langle f, \tilde{\pi}(t,a,b)g\rangle = 0$ for all $(t,a,b) \in \tilde{H}_n$. Then $\|f\|_2 \|g\|_2 = 0$ and it follows that $f = 0$. Hence $\tilde{\pi}$ is irreducible as desired. $\square$

Proposition 6.3. *Let $n \in \mathbb{N}$, let $(a_k, b_k) \in \mathbb{R}^{2n}$ be distinct nonzero elements such that (a_k, b_k) generate a discrete subgroup of $\mathbb{R}^{2n}$ and $a_h \cdot b_k \in \mathbb{Q}$ for all h, k (where $k, h \in \mathbb{N}$). If $r \in \mathbb{N}$ and*

$$\sum_{k=1}^{r} c_k e^{2\pi i b_k \cdot t} f(t + a_k) = 0$$

with $0 \neq c_k \in \mathbb{C}$ constants, then $f = 0$.

Proof. We have $\mathbb{R}^{2n} = \tilde{H}_n/\mathbb{T}$. Lift the (a_k, b_k) to the elements $g_k := (1, a_k, b_k) \in \tilde{H}_n$. Note that the hypothesis $a_h \cdot b_k \in \mathbb{Q}$ ensures that $\langle g_1, \dots, g_r\rangle$ is a discrete subgroup of $\tilde{H}_n$. We claim that if $0 \neq d_k \in \mathbb{C}$, then $\alpha := \sum_{k=1}^{r} d_k g_k$ is a nonzerodivisor in $\mathbb{C}\tilde{H}_n$. Indeed if $0 \neq \beta \in \mathbb{C}\tilde{H}_n$ and $\alpha\beta = 0$, let T be a transversal for $\mathbb{T}$ in $\tilde{H}_n$ containing $\{g_1, \dots, g_r\}$ and write $\beta = \sum_{t \in T} \beta_t t$ where $\beta_t \in \mathbb{C}\mathbb{T}$. Since $\mathbb{R}^{2n}$ is an ordered group, we can apply a leading term argument: let k be such that $g_k \in T$ is largest and let $s \in T$ be the largest element such that $\beta_s \neq 0$. Then by considering $g_k s$, we see that $\alpha\beta \neq 0$ because $d_k \beta_s \neq 0$, which is a contradiction. The result now follows from Proposition 1.5, Corollary 6.2 and Theorem 5.2. □

7. Shearlet Groups

We now investigate the problem of linear independence of left translations of functions in $L^2(S)$, where S denotes the shearlet group. This fits the theme of our paper since S has an irreducible, square integrable representation on $L^2(\mathbb{R}^2)$. We begin by defining the shearlet group.

For $a \in \mathbb{R}^+$ (the positive real numbers) and $s \in \mathbb{R}$ let $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$, $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and let $G = \{S_s A_a \mid a \in \mathbb{R}^+, s \in \mathbb{R}\}$. The *shearlet group* S is defined to be $S = G \ltimes \mathbb{R}^2$. The group multiplication for S is given by $(M, t)(M', t') = (MM', t + Mt')$, where $M \in G$ and $t \in \mathbb{R}^2$ (here we are considering elements of $\mathbb{R}^2$ as column vectors). The left Haar measure for S is $\frac{dadsdt}{a^3}$ and the right Haar measure for S is $\frac{dadsdt}{a}$, so S is a nonunimodular group. A representation π of S on $L^2(\mathbb{R}^2)$ can be defined by

$$\pi(S_s A_a, t) f(x) = a^{-3/4} f((S_s A_a)^{-1}(x - t)).$$

The representation π is square integrable and irreducible, see [4, §2] for the details. We shall write f_{ast} to indicate $\pi(S_s A_a, t)f$. The function f_{ast} is also known as the *shearlet transform* of f. Since the shearlet transform is realized by an irreducible, square integrable representation of S on $L^2(\mathbb{R}^2)$, the question of linear independence of the left translations of a function in $L^2(S)$ is related to the question of the linear independence of the shearlets of a function in $L^2(\mathbb{R}^2)$. The question of linear independence of the shearlet transforms of f now becomes: Is $f = 0$ the only solution in $L^2(\mathbb{R}^2)$ that satisfies

$$\sum_{k=1}^{n} c_k f_{a_k s_k t_k} = 0 \tag{7.1}$$

where c_k are nonzero constants and $(a_k, s_k, t_k) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$?

Proposition 7.2. *Let S be the shearlet group. There exists a nonzero function in $L^2(S)$ that has linearly dependent left translations.*

Proof. The proposition will follow immediately from Proposition 1.5 if we can show there exists a nonzero $f \in L^2(\mathbb{R}^2)$ that satisfies (7.1), which we now do. Combining [13, Theorem 4.6] and [9, Example 5] we see that there exists a continuous nonzero $f \in L^2(\mathbb{R}^2)$ that satisfies

$$f(x) = \sum_{\beta \in \mathbb{Z}^2} a(\beta) f(A_4^{-1}x - \beta), \tag{7.3}$$

where $a(\beta) \in \mathbb{C}\mathbb{Z}^2$ and $x \in \mathbb{R}^2$. The function f is said to be *refinable*. In the literature $a(\beta)$ is often referred to as a mask. The important thing here is that $a(\beta)$ has finite support. If $\beta = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then set $\beta' = \begin{pmatrix} 4b_1 \\ 2b_2 \end{pmatrix}$. Using (7.3) we obtain

$$\begin{aligned}
\pi(S_0 A_1, 0) f(x) &= \sum_{\beta \in \mathbb{Z}^2} a(\beta) 4^{3/4} 4^{-3/4} f(A_4^{-1}x - \beta) \\
&= \sum_{\beta \in \mathbb{Z}^2} a(\beta) 4^{3/4} 4^{-3/4} f(A_4^{-1}x - A_4^{-1}\beta') \\
&= \sum_{\beta \in \mathbb{Z}^2} 4^{3/4} a(\beta) \pi(S_0 A_4, \beta') f(x) \\
&= \sum_{\beta \in \mathbb{Z}^2} 4^{3/4} a(\beta) f_{4,0,\beta'}(x).
\end{aligned}$$

Hence, there is a linear dependency among the shearlet transforms of f, proving the proposition. □

Remark 7.4. *The refinable function f used in the proof of the previous proposition has compact support since $a(\beta)$ has finite support* [9, Theorem 5]. *Compare this to* [19, Theorem 4.3] *where it was shown, in a slightly different setting, that a compactly supported separable shearlet system is linearly independent. Thus it appears that in general the hypothesis of separability is important.*

The next result gives a sufficient condition for linear independence of a shearlet system.

Proposition 7.5. *Let $0 \neq f \in L^2(\mathbb{R}^2)$. Then $\{f_{1nt} \mid n \in \mathbb{Z}, t \in \mathbb{Z}^2\}$ is a linearly independent set.*

Proof. Let $H = \{S_n A_1 \mid n \in \mathbb{Z}\}$, then $K = H \ltimes \mathbb{Z}^2$ is a torsion-free discrete subgroup of S. Because H and $\mathbb{Z}^2$ are solvable, K is solvable and thus satisfies the strong Atiyah conjecture. By Theorem 1.6 the K-left translations of a function in $L^2(S)$ are linearly independent. The proposition now follows from Proposition 1.5. □

The results obtained in this section are similar to the results from Section 3 for the affine group. This is not surprising since the shearlet transform involves a dilation and a translation.

8. Virtually abelian groups

In this section we consider virtually abelian groups, that is groups with an abelian subgroup of finite index.

Proposition 8.1. *Let G be a locally compact group which has an abelian closed subgroup A of finite index, and let $1 \leq p \in \mathbb{R}$. Assume that if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$. Let $0 \neq f \in L^p(G)$, let $H \leqslant G$ and let $\theta \in \mathbb{C}H$.*

(a) *If θ is a nonzero divisor in $\mathbb{C}H$, then $\theta f \neq 0$.*
(b) *If H is torsion free and $\theta \neq 0$, then $\theta f \neq 0$.*

Proof. Note that $\mathbb{C}A$ is an integral domain. Let B be the intersection of the conjugates of A in G, then B is a closed abelian normal subgroup of finite index in G. Let $\{a_1, \dots, a_m\}$ be a set of coset representatives for B in A. Then $L^p(A) = \bigoplus_{i=1}^m L^p(B)a_i$ and we see that if $0 \neq \phi \in \mathbb{C}B$ and $0 \neq f \in L^p(B)$, then $\phi f \neq 0$. Let $\{g_1, \dots, g_n\}$ be a set of coset representatives for B in G. Then $L^p(G) = \bigoplus_{i=1}^n L^p(B)g_i$. We may view this as an isomorphism of $\mathbb{C}B$-modules. Set $S = \mathbb{C}B \setminus \{0\}$. Then we may form the ring of fractions $S^{-1}\mathbb{C}G$. Since every element of S is a non-zerodivisor in $\mathbb{C}G$, it follows that $S^{-1}\mathbb{C}G$ is a ring containing $\mathbb{C}G$. Furthermore $S^{-1}\mathbb{C}B$ is a field, and $S^{-1}\mathbb{C}G$ has dimension n over this field. Therefore $S^{-1}\mathbb{C}G$ is an artinian ring, and since $S^{-1}\mathbb{C}B$ is a field of characteristic zero, we see that $S^{-1}\mathbb{C}G$ is a semisimple artinian ring, by Maschke's theorem. We deduce that non-zerodivisors in $S^{-1}\mathbb{C}G$ are invertible. Using [8, Theorem 10.8], we may form the $S^{-1}\mathbb{C}G$-module $S^{-1}L^p(G)$.

(a) If θ is non-zerodivisor in $\mathbb{C}H$, then θ is a non-zerodivisor in $\mathbb{C}G$ and hence is invertible in $S^{-1}\mathbb{C}G$, so θ^{-1} exists. We may regard f as an element of $S^{-1}L^p(G)$, because $S^{-1}L^p(G)$ contains $L^p(G)$. So if $\theta f = 0$, then $\theta^{-1}\theta f = 0$, consequently $f = 0$ and we have a contradiction.
(b) If H is torsion free, then we know that every non-zero element of $\mathbb{C}H$ is a non-zerodivisor in $\mathbb{C}H$; this was first proved by K. A. Brown [2]. Thus the result follows from (a). □

We now use the previous result to give the following generalization of [5, Theorem 1.2].

Theorem 8.2. *Let G be a locally compact group with no nontrivial compact subgroups, and suppose G has an abelian closed subgroup of finite index. Then every nonzero element of $L^p(G)$, where $1 \leq p \leq 2$, has linearly independent translations.*

Proof. Since G has no nontrivial compact subgroups, it is torsion free. Furthermore for $1 \leq p \leq 2$, if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$ by [5, Theorem 1.2]. The result now follows from Proposition 8.1(b). □

Theorem 8.3. *Let G be a locally compact abelian group, let $n \in \mathbb{N}$, and let $1 \leq p \in \mathbb{R}$. Assume that $p \leq 2n/(n-1)$. Suppose G has a closed subgroup of finite index isomorphic to $\mathbb{R}^n$ or $\mathbb{Z}^n$ as a locally compact abelian group. Let $H \leqslant G$, let $\theta \in \mathbb{C}H$, and let $\theta \in \mathbb{C}G$ and let $0 \neq f \in L^p(G)$.*

(a) If θ is a nonzero divisor in $\mathbb{C}H$, then $\theta f \neq 0$.
(b) If H is torsion free and $\theta \neq 0$, then $\theta f \neq 0$.

Proof. We apply Proposition 8.1 with $A = \mathbb{R}^n$ or $\mathbb{Z}^n$. We need to check the hypothesis that if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$. For the case $A = \mathbb{R}^n$, this follows from [22, Theorem 3], while for the case $A = \mathbb{Z}^n$, this follows from [17, Theorem 2.1]. □

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